TOPOLOGIZED FUNDAMENTAL GROUPS

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Abstract. We introduce the natural quotient topology on π_1, with a focus on the Hawaiian earring group.

Note: All diagrams have been sourced from the Wild Topology blog, by J. Brazas [3].

1. Motivation

We begin by recalling one of the most important theorems in algebraic topology.

Definition 1.0.1 (Semilocally simply-connected)**.** We say that X is semilocally simply-connected if each $x_0 \in X$ admits a neighborhood U such that $\pi_1(U, x_0) \to \pi_1(X, x_0)$ is trivial. This ensures that X has a simply-connected cover.

Theorem 1.0.2 (Galois correspondence)**.** Suppose X is path-connected, locally path-connected, and **semilocally simply-connected**. Then for every $H \leq \pi_1(X, x_0)$ there is a covering $p : X_H \to X$ such that $p_*\pi_1(X_H, \widetilde{x}_0) = H$ for a suitable choice of $\widetilde{x}_0 \in p^{-1}(x_0)$.

Example 1.0.3 (Hawaiian earring). Let

$$C_n = \left\{(x, y) \in \mathbb{R}^2 \mid (x - 1/n)^2 + y^2 = 1/n^2\right\}.$$

The Hawaiian earring space is defined as $\mathbb{E} := \bigcup_{n\geq 1} C_n$, with the subspace topology coming from $\mathbb{R}^2$.
The space $\mathbb{E}$ is not the same as $\bigvee_{\mathbb{N}} S^1$; the topology of $\bigvee_{\mathbb{N}} S^1$ is finer than that of $\mathbb{E}$.
We define the earring group as $\pi_1(\mathbb{E})$.

It is a straightforward observation that $\mathbb{E}$ is **not** semilocally simply-connected: neighborhoods of the accumulation point necessarily contain all but finitely many of the C_n's.

Moreover, classical algebraic topology cares only about the algebra, and does not keep track of, for example, the convergence of the C_n's to a point: we need to topologize π_1 to record that information. In order to study the algebraic topology of wild spaces like $\mathbb{E}$, we introduce an enriched version of π_1 that remembers more of the topology of the underlying space.

2. Inverse limits

Definition 2.0.1 (Inverse limit)**.** An inverse system of groups/spaces is a pair $((A_i), (f_{ji}))_{i,j\in I}$ (with $(I, \leq)$ a poset), where (A_i) is an indexed family of groups/spaces and morphisms $f_{ji} : A_j \to A_i$ for all $i \leq j$, such that $f_{ii} = \mathrm{Id}$ and $f_{ki} = f_{ji} \circ f_{kj}$ for all $i \leq j \leq k$.

$$\cdots \to A_3 \xrightarrow{f_{32}} A_2 \xrightarrow{f_{21}} A_2.$$

The inverse limit of $((A_i),(f_{ji}))_{i,j\in I}$ is

$$A = \varprojlim_{i\in I} A_i := \left\{ \vec{a} \in \prod A_i \mid a_i = f_{ji}(a_j) \text{ for all } i \leq j \right\}.$$

In the category of groups, A is a group with pointwise multiplication. In the category of topological spaces, A is a topological space endowed with the subspace topology from the product topology.

Remark 2.0.2. The inverse limit comes equipped with natural projections $A \to A_i$.
The subspace topology on an A coincides with the **initial topology**: the coarsest topology that makes every $A \to A_i$ continuous.

3. The natural quotient topology

Consider the based loop space $\Omega(X, x_0)$, and endow it with the compact-open topology. We have a projection $q : \Omega(X, x_0) \to \pi_1(X, x_0)$ (the π_0 map), so we assume $\pi_1(X, x_0)$ to have the quotient topology with respect to q.

Let $r : \Omega(X, x_0) \to \Omega(X, x_0)$ be the reverse map $\alpha \mapsto \alpha^{-1}$. Letting $\overline{r}$ be the corresponding map on π_1, we have the commutative square

$$\begin{array}{ccc} \Omega(X,x_0) & \xrightarrow{r} & \Omega(X,x_0) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \pi_1(X,x_0) & \xrightarrow{\overline{r}} & \pi_1(X,x_0) \end{array}$$

Note that q is quotient and r is continuous. In particular, $q \circ r$ is continuous, $\overline{r}$ is continuous by the universal property of quotient maps. Thus inversion is continuous in $\pi_1(X, x_0)$.

On the other hand, let c be the concatenation map in $\Omega(X, x_0)$, and $\mu : \pi_1(X, x_0) \to \pi_1(X, x_0)$ be multiplication. Then the following square commutes.

$$\begin{array}{ccc} \Omega(X,x_0)\times\Omega(X,x_0) & \xrightarrow{c} & \Omega(X,x_0) \\ {\scriptstyle q\times q}\downarrow & & \downarrow{\scriptstyle q} \\ \pi_1(X,x_0)\times\pi_1(X,x_0) & \xrightarrow{\mu} & \pi_1(X,x_0) \end{array}$$

FALSE PROOF: Since q is a quotient map, then $q \times q$ is quotient. But c is continuous, hence $q \circ c$ is continuous, and μ is continuous by the universal property of quotient maps.

Observation 3.0.1. The product of quotient maps need not be a quotient map. Open maps are quotient, and the product of open maps is open, so we do have a topological group when the projection q is open.
In general, the natural quotient topology is a quasitopological group, but it's not always a topological group.

Example 3.0.2. Let G be a topological group. Then the based loop space $\Omega(G, e)$ with the compact-open topology is a topological group of the product of loops $(\gamma_1 \cdot \gamma_2)(t) = \gamma_1(t)\gamma_2(t)$ (loop concatenation does not give a group structure on $\Omega(G, e)$). It is well-known that the projection $H \to \pi_0 H$ from a topological group is an open map, so $q : \Omega(G, e) \to \pi_1(G, e)$ is

an open map, and by the Eckmann-Hilton argument, multiplication in $\Omega(G, e)$ descends to the usual product in π_1.

Remark 3.0.3.

- We can also construct $\mathbb{E}$ as an inverse limit, by letting $X_n = \bigcup_{k=1}^{n} C_k$. If $n > m$, then there is a retraction $r_{n,m} : X_n \to X_m$ which collapses the circles C_k for $m < k \leq m$. We then have an inverse system
$$\cdots \to X_{n+1} \xrightarrow{r_{n+1,n}} X_n \xrightarrow{r_{n,n-1}} \cdots \xrightarrow{r_{3,2}} X_2 \xrightarrow{r_{2,1}} X_1.$$
- Similarly, we have retractions $r_n : \mathbb{E} \to X_n$ by collapsing $\bigcup_{k=n+1}^{\infty} C_k$ (which is actually a homeomorphic copy of $\mathbb{E}$). With the initial topology, $\varprojlim_n X_n \approx \mathbb{E}$. This follows from a few observations: the maps r_n give us a continuous map $\mathbb{E} \to \varprojlim_n X_n$ by the universal property of inverse limits which can be checked to be bijective. But $\mathbb{E}$ is compact and $\varprojlim_n X_n$ is Hausdorff as a subspace of a product of Hausdorff spaces, and any continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

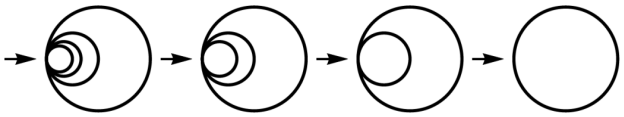


FIGURE 1. Inverse system $\{r_{n,m}\}$

We apply the π_1 functor to obtain the diagram,

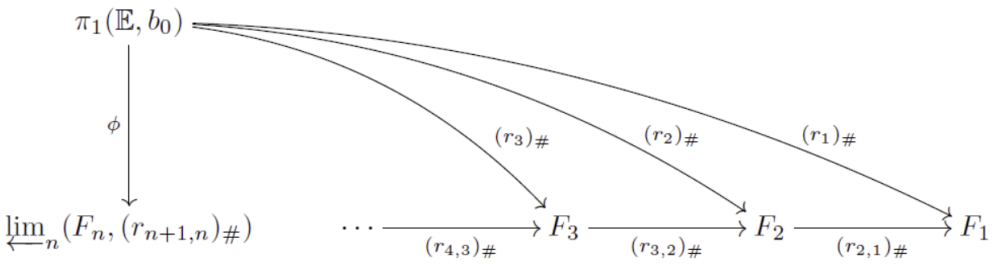


FIGURE 2. Enter Caption

where $F_n \cong \pi_1(X_n)$ is generated by $a_1, a_2, \ldots, a_n$.

Proposition 3.0.4. $\pi_1(\mathbb{E})$ is uncountable.

This follows because we can construct loops $\alpha : [0, 1] \to \mathbb{E}$ by winding around C_1 in $[0, 1/2]$, then around C_2 in $[1/2, 3/4]$, around C_3 in $[3/4, 7/8]$ and so on, which is continuous at $t = 1$ because $\operatorname{diam}(C_n) \to 0$. Let $I_n = [(2^n - 1)/2^n, (2^{n-1} - 1)/2^{n-1})]$ (starting at $n = 0$). Then, to any binary sequence $d = (d_n) \in \{0, 1\}^{\mathbb{N}}$ we can assign a loop α_d, where $\alpha_d|_{I_n}$ wraps around C_n if $d_n = 1$, and it's constant at the basepoint if $d_n = 0$.

Proposition 3.0.5. The induced homomorphism $\phi : \pi_1(\mathbb{E}) \to \varprojlim_n F_n$ is injective. (but it is not a topological embedding)

This is the Shape Injectivity Theorem, which we will not prove. The proof involves the study of the topology of the universal covers $p_n : \widetilde{X}_n \to X_n$, lifts of the retractions $r_{n+1,n} : X_{n+1} \to X_n$ to these covers, and the limit $\varprojlim_n \widetilde{X}_n$, which surjects onto $\varprojlim_n X_n$ via $\varprojlim_n p_n$.

One may be tempted to try to show that $\pi_1(\mathbb{E}) \cong \varprojlim_n \pi_1(X_n)$ because $\mathbb{E} \approx \varprojlim_n X_n$, but this is simply not true.

Proposition 3.0.6. ϕ is not surjective.

Proof. Consider the sequence $(w_n) \in \varprojlim_n \pi_1(X_n)$ given by $w_n = \prod_{i=1}^{n}[a_1, a_{i+1}]$. Note that a_1 appears infinitely many times in (w_n), but no continuous loop in $\pi_1(\mathbb{E})$ can wrap around C_1 infinitely many times. ∎

Lemma 3.0.7. $\pi_1(\mathbb{E})$ *is Hausdorff.*

Proof. The natural quotient topology on F_n is discrete (in particular, Hausdorff), hence $\prod_n F_n$ is Hausdorff (Hausdorffness is product-closed), and $\varprojlim_n F_n$ inherits Haudorffness. By injectivity and continuity of $\phi : \pi_1(\mathbb{E}) \to \varprojlim_n F_n$, whence $\pi_1(\mathbb{E})$ is Hausdorff. ∎

Proposition 3.0.8. The product $q \times q$ is not a quotient map, μ is discontinuous, and $\pi_1(\mathbb{E})$ is not a topological group.

Let $x_i = [\ell_i]$, where ℓ_i is the usual parametrization of C_i in $\mathbb{E}$. Define $w_{mn} = [x_m, x_n]^m$, and $v_{nm} = [x_1, x_m]^n$. Let $W = \{w_{mn} \mid m, n > 1, m \neq n\}$, and $V = \{v_{mn} \mid m, n > 1, m \neq n\}$. Lastly, let $Z = VW$. We observe that $\mu^{-1}(Z)$ is not closed in $G \times G$:

(1) Fix m. Then the sequence $(w_{mn})_n$ converges to e.
(2) Fixing n, note that $(v_{m,n})_m$ converges to e.
(3) Since $e \notin Z$, then $(e, e) \notin \mu^{-1}(Z)$.
(4) Since e is a limit point of both V and W, then (e, e) is a limit point of $V \times W \subset \mu^{-1}(Z)$. Therefore (e, e) is a limit point of $\mu^{-1}(Z)$ not in Z.

Lemma 3.0.9. *Let X be a Hausdorff sequential space, let Y be Hausdorff, and $q : X \to Y$ be a quotient map. Then every convergent sequence in $(y_n) \to y$ lifts to a sequence (x_n) with a convergent subsequence $(x_{n_k}) \to x$, such that $q(x_{n_k}) = y_{n_k}$ and $q(x) = y$.*

Proposition 3.0.10. Z is closed in G.

Proof. Let $(g_k) = (w_{m_k n_k} v_{m_k n_k}) \subset Z$ be a sequence that converges to $g \in \pi_1(\mathbb{E})$. We want to show that $g \in Z$.
By the Lemma, we have a sequence $(\alpha_j) \to \alpha$ such that $[\alpha_j] = g_{k_j}$ and $[\alpha] = g$. We claim that the sequences $\{m_{k_j}\}$ and $\{n_{k_j}\}$ are bounded.
Suppose that $\{n_{k_j}\}$ is unbounded. Then the loop α_j winds around C_1 at least $2n_{k_j}$ many times, so (α_j) cannot converge uniformly, contradiction.
Suppose now that $\{n_{k_j}\}$ is bounded. If $\{m_{k_j}\}$ is unbounded, then by the Pigeon-Hole principle, the sequence $\{n_{k_j}\}$ takes on a value N infinitely many values of j. We pass to the subsequence of all k_j such that $N = n_{k_j}$ (i.e. assume $n_{k_j} = N$). Since α_j represents $[x_{m_{k_j}}, x_N]^{m_{k_j}}[x_1, x_{m_{k_j}}]^N$, then it must wind around C_N at least $(2m_{k_j})$ many times, so (α_j) cannot converge uniformly.
Then both $\{m_{k_j}\}$ and $\{n_{k_j}\}$ are bounded, so the subsequence (g_{k_j}) can only take on finitely many values. But G is Hausdorff, hence (g_{k_j}) is eventually constant, thus $g \in Z$. ∎

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